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A HOMOGENEITY CONDITION GUARANTEEING BIFURCATION IN MULTIPARAMETER NONLINEAR EIGENVALUE PROBLEMS

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1. INTRODUCTION

THE BIFURCATION theorem of Krasnosel'skii [18] concerns the equation

$$e = \lambda Ae + H(\lambda, e), \tag{1.1}$$

where e is an element of a real Banach space E, λ is a real scalar, $A : E \to E$ is compact and linear, and $H : \mathbb{R} \times E \to E$ is compact, continuous, and higher order in u. It states that if $\lambda \in \mathbb{R}/\{0\}$ is a characteristic value of A (i.e. $1/\lambda$ is an eigenvalue for A) of odd algebraic multiplicity, then $(\lambda, 0)$ is a branch (or bifurcation) point for the solution set to (1.1). This result and its global extension, due to Rabinowitz [20], have had a significant impact in mathematical research in recent years, fostering advances, not only in the use of topological methods in nonlinear analysis, but also in differential equations and its applications.

One direction in which research has prospered is in the study of multiparameter nonlinear eigenvalue problems (i.e. problems in which the parameter is taken from a vector space of dimension >1). Among the numerous examples of such are the work of Alexander and Antman [1], [2]; Alexander and Fitzpatrick [3], [4]; Antman and Keeney [5]; Browne and Sleeman [6], [7]; Cantrell [8]–[10]; Hale [15] and Turyn [23].

For the purpose of indicating the thrust of this article, we very briefly note the results of several of the above mentioned works. In [9] and [10], the author studies the equation

$$e = \sum_{i=1}^{k} \lambda_i A_i e + H(\lambda, e), \tag{1.2}$$

where $\lambda = (\lambda_1, \ldots, \lambda_k)$, with conditions on the operators analogous to those in (1.1) in order to analyze the solution set to the nonlinear Klein oscillation problem (see also [6], [7], [13], and [16]). The methods are primarily degree theoretic in a manner analogous to [18].

In [1] and [2], Alexander and Antman use cohomological techniques to assert the multi-dimensionality of bifurcating solution continua for appropriate analogues to (1.1), including problems in which the parameter space is of infinite dimension. These methods have been applied to the study of the buckling of nonlinearly elastic rods under torsion, thrust and gravity [4].

The existence of parameter values for which the topological indices associated with (1.1) differ is the crucial aspect in the proof of Krasnosel'skii's Bifurcation Theorem. The same is true for the examples mentioned above. In [10], for example, the form of the linearization of (1.2) is such that rays in $\mathbb{R}^k$ emanating from the origin provide paths along which one may anticipate a change of topological index. However, such paths may not always be guaranteed to exist for the operator equation

$$e = A(\lambda)e + H(\lambda, e), \tag{1.3}$$

where $\lambda \in \Lambda$, an arbitrary (real) Banach space. The principal objective of this paper is then to observe a readily verifiable condition on the map $\lambda \to A(\lambda)$, which guarantees the existence of paths along which odd algebraic multiplicity implies a change of index. Once this observation, homogeneity condition (2.2), is made, Krasnosel'skii and Rabinowitz type bifurcation theorems are immediate, and are collected in Section 2.

If $\dim \Lambda > 1$, there are some noteworthy consequences of (2.2) upon the structure of the set of bifurcation points in Λ. In particular, in Section 3 we note that a bifurcation point of odd multiplicity is never isolated in the set of bifurcation points. We also introduce a notion of "codimension 1 manifolds" and use this concept and a perturbation theory result to completely describe the set of bifurcation points near a point of multiplicity one, under the assumption that the map $\lambda \to A(\lambda)$ is continuous.

Finally, in Section 4, we demonstrate that a class of nonlinear Sturm–Liouville boundary value problems with coefficient functions viewed as parameters satisfies (2.2), and use this information to analyze bifurcation phenomena. The analysis here, combined with that for a similar example in [2], gives a rather complete description of the solution set to such problems.

2. SITUATION AND MAIN RESULTS

Suppose E and Λ are real Banach spaces and that $D \subseteq \Lambda$ is a nonempty open set with the property that $tD \subseteq D$ for $t > 0$. Let $F: D \times E \to E$ be a map such that $F|_{\Gamma \times E}$ is completely continuous for compact subsets Γ of D. More specifically assume $F(\lambda, e) = A(\lambda)e + H(\lambda, e)$, where $A(\cdot): D \to K(E)$ (the Banach space of compact linear operators on E) is continuous and $H(\lambda, e)/\|e\| \to 0$ as $\|e\| \to 0$ uniformly for λ contained in compact subsets of D. Consider the solution set in $D \times E$ of the equation

$$e = F(\lambda, e). \tag{2.1}$$

Note that $(\lambda, 0)$ is a solution to (2.1) for all values λ in D. Such solutions are called *trivial*, and by a *nontrivial* solution to (2.1), we mean a solution (λ, e) with $e \neq 0$. B will then denote the set $\{\lambda \in D$: every neighborhood of $(\lambda, 0)$ in $D \times E$ contains a nontrivial solution to (2.1)$\}$. $B \times \{0\}$ is said to be the set of bifurcation points for (2.1).

Let $\Sigma_A = \{\lambda \in D$: the null space $N(I - A(\lambda))$ is nontrivial in $E\}$. Then it is a basic result that B and Σ_A are closed in D, and $B \subseteq \Sigma_A$. Furthermore, if $\lambda \in \Sigma_A$, we let geomult $\lambda = \dim N(I - A(\lambda))$ and mult $\lambda = \dim \bigcup_{r \geq 1} N\{[I - A(\lambda)]^r\}$.

For the remainder of this paper we also assume:

$$A(t\lambda) = t^k A(\lambda), \tag{2.2}$$

for all $t > 0$, $\lambda \in D$ and for some fixed $k \in Z \backslash \{0\}$. We then have:

THEOREM 2.1. If $\lambda \in \Sigma_A$ and mult λ is odd, then $\lambda \in B$.

Remarks 2.2. (i) If $\dim \Lambda = 1$, $k = 1$ in (2.2), and $\lambda \to A(\lambda)$ is an odd map, theorem 2.1 is the classical Krasnosel'skii Bifurcation Theorem. Thus theorem 2.1 may be viewed as a multiparameter analogue to the Krasnosel'skii result. The proof of theorem 2.1 will proceed as in [18] once an appropriate change of topological index lemma is established. For the sake of completeness, we give an explicit statement of this lemma.

LEMMA 2.3. Suppose (2.2) holds and $\lambda \notin \Sigma_A$. Then the Leray–Schauder topological degree $\deg_{LS}(I - A(\lambda), B(0, R), 0)$ is well-defined and equals

$$\sum_{(-1)i=1}^{m} \beta_i$$

where $\beta_i = \text{mult}(t_i^{1/k}\lambda)$, with $\{t_i : i = 1, 2, \ldots, m\}$ the finite number of points in (0, 1) such that $t_i^{1/k}\lambda \in \Sigma_A$.

(ii) If $\Lambda = \mathbb{R}^k$ and $A(\lambda) = \sum_{i=1}^{k} \lambda_i A_i$, where $\lambda = (\lambda_1, \ldots, \lambda_k)$ and $A_i \in K(E)$, then (2.2) holds with $k = 1$. Hence the results in [9] and [10] are also special cases of the results of this paper.

(iii) If (2.2) holds for some k, $0 \notin D$ if $k < 0$ and $0 \notin \Sigma_A$ if $k > 0$.

(iv) In [1] and [2], Alexander and Antman use cohomological results to establish global higher dimensional branches of solutions to multiparameter nonlinear eigenvalue problems which emanate from bifurcation points. They essentially assume the existence of λ_1, λ_2 in D and an arc in D joining λ_1 and λ_2 such that $\deg_{LS}(I - A(\lambda_1), B(0, R),) \neq \deg_{LS}(I - A(\lambda_2), B(0, R), 0)$. (2.2) provides a method of verifying this condition and hence identifies a class of problems to which the techniques of [1] and [2] are applicable.

Definition 2.4. A continuous map $h : \mathbb{R} \to D$ is a *proper crossing of changing degree at* λ if the following conditions hold:

(i) $h(0) = \lambda$ and $h(t) \to \partial D$ as $t \to \pm\infty$;

(ii) if $\gamma > 0$, there is a neighborhood V of λ in D such that $h^{-1}(V \cap h(\mathbb{R})) \subseteq (-\gamma, \gamma)$;

(iii) there is a number $\varepsilon_h > 0$ such that

(a) the Leray–Schauder degree $\deg_{LS}(I - A(h(t)), B(0, 1), 0)$ is defined for all t such that $|t| < \varepsilon_h$ and $t \neq 0$; and;

(b) $\deg_{LS}(I - A(h(\tau)), B(0, 1), 0) = \text{sgn}(\tau\beta) \cdot \deg_{LS}(I - A(h(\beta)), B(0, 1), 0)$, where $\tau, \beta \in (-\varepsilon_h, \varepsilon_h)$, $\tau \neq 0$, $\beta \neq 0$.

Suppose now that $\lambda \in \Sigma_A$. By (2.2) there is a $\delta > 0$ such that $([1 - \delta, 1 + \delta]\lambda) \cap \Sigma_A = \{\lambda\}$. Since Σ_A is closed, for each $\sigma \in [1 - \delta, 1) \cup (1, 1 + \delta]$ there is an $\varepsilon(\sigma\lambda) > 0$ such that $B(\sigma\lambda, \varepsilon(\sigma\lambda)) \subseteq D \backslash \Sigma_A$. Thus there is a "cone" Υ at λ which contained in $D \backslash \Sigma_A$. We now have the following theorem.

THEOREM 2.5. Suppose $h : \mathbb{R} \to D$ is a *proper crossing of changing degree at* λ. Then there is a continuum (closed connected set) $\mathscr{C}$ of nontrivial solutions to (2.1) meeting $(\lambda, 0)$ such that either

(i) $\mathscr{C}$ approaches $\partial(D \times E)$

or

(ii) $\mathscr{C} \cap (B \times \{0\} \backslash \{(\lambda, 0)\}) \neq \varnothing$.

Furthermore, if $\pi : D \times E \to D$ is the projection map, then $\pi(\mathscr{C}) \subseteq h(\mathbb{R})$.

Proof. Note that by lemma 2.3 and the above exposition, proper crossings of changing degree exist if mult λ is odd. Define a map $G_h: \mathbb{R} \times E \to E$ by $G_h(t, e) = A(h(t))e + H(h(t), e)$. By the properties of h, the result follows from an application of the Rabinowitz Bifurcation Theorem [20] to the equation $e = G_h(t, e)$.

Remark 2.6. A version of this result, where a change of index is assumed is mentioned in [1]. However, a condition such as (2.2) is needed in order to have the result follow from the assumption of odd algebraic multiplicity at λ.

3. AN EXAMINATION OF B

Let $\tilde{D}$ be a component of D. Since Λ is a Banach space, D is locally path connected. Thus $\tilde{D}$ is path connected, open, and $t\tilde{D} \subseteq \tilde{D}$ for $t > 0$. It then follows from the homotopy invariance of the Leray–Schauder topological degree that if $\lambda \in \Sigma_A \cap \tilde{D}$ and mult λ is odd, $\tilde{D}\backslash\Sigma_A$ is not connected. Hence, in this situation, Σ_A must have topological dimension $\dim \Lambda - 1$ if $\dim \Lambda < \infty$ and dimension ∞ if $\dim \Lambda = \infty$.

The preceding remarks indicate that, if $\dim \Lambda > 1$, the set B is substantially more than a discrete collection of points (the situation if $\dim \Lambda = 1$). In this section, we make an investigation into the *a priori* structure of B. We first make a definition which shall prove useful.

Definition 3.1. Let E be a real Banach space. A subset V of E is called a codimension 1 manifold if for every $x \in V$ there is a positive number $r(x)$ such that $B(x, r(x)) \cap V$ is homeomorphic to a proper basic neighborhood in $\tilde{E} = \{x \in E : \|x\| = 1\}$.

We now have the following basic result.

THEOREM 3.2. If Σ_A at λ is a codimension 1 manifold, say T, and mult λ is odd, then $T \subseteq B$.

THEOREM 3.3. Let $B' = \{\mu \in \Sigma_A$: mult μ is odd$\}$. Let $\lambda \in B'$ and suppose there is $\eta > 0$ such that $\overline{B(\lambda, \eta)} \cap B'$ is closed. Then there is a codimension 1 manifold $M \subseteq \overline{B(\lambda, \eta)} \cap B'$ containing λ.

Proof. By remark 2.2(iii), we may pick $d > 0$ such that $d < \|\lambda\|$, $B(\lambda, d) \subseteq D$, and $\mathbb{R}\{\lambda\} \cap \Sigma_A \cap B(\lambda, d) = \{\lambda\}$. Then pick $\delta \in (0, 1)$ such that $(1-\delta)\lambda$, $(1+\delta)\lambda \in B(\lambda, d)$. Let $\lambda_- = (1-\delta)\lambda$ and $\lambda_+ = (1+\delta)\lambda$. Then there is $\varepsilon > 0$ such that $B(\lambda_\sigma, \varepsilon) \subseteq B(\lambda, d) \cap (D\backslash\Sigma_A)$ for $\sigma = -, +$. Note that if $\|\mu - \lambda_+\| < \varepsilon$, then

$$\left\| \frac{(1-\delta)}{(1+\delta)}\mu - \lambda_- \right\| < \varepsilon.$$

Hence

$$\deg_{LS}(I - A(\lambda_-), B(0, 1), 0) = \deg_{LS}\left[I - A\left(\frac{(1-\delta)}{(1+\delta)}\mu\right), B(0, 1), 0\right]$$

and

$$\deg_{LS}(I - A(\lambda_+), B(0, 1), 1) = \deg_{LS}(I - A(\mu), B(0, 1), 0).$$

Since mult λ is odd, there is

$$t \in \left(\frac{1-\delta}{1+\delta}, 1\right)$$

such that mult $(t\mu)$ is odd.

Now let d, δ, and ε be as above with the additional restrictions that

$$\delta \leqslant \frac{\eta}{2(\|\lambda\|+1)} \qquad \text{and} \qquad \varepsilon < 2\delta.$$

Let

$$C = B\left(\lambda, \frac{\varepsilon}{2}\right) \cap \{\mu \in \Lambda : \|\mu\| = \|\lambda\|\}.$$

Then C is a codimension 1 manifold. Let $\mu \in C$. Then $(1\sigma\delta)\mu \in B(\lambda_\sigma, \varepsilon)$, $\sigma = -, +$. Thus there is $t \in (1-\delta, 1+\delta)$ such that mult$(t\mu)$ is odd. Now define a function

$$f : C \to (1-\delta, 1+\delta) \qquad \text{by} \qquad f(\mu) = \min_{t \in (1-\delta, 1+\delta)} \{t : \text{mult}(t\mu) \text{ is odd}\}.$$

We now show f is continuous.

Suppose $\{\mu_n\}_{n=1}^\infty \subseteq C$ and $\mu_n \to \mu$. Let $\{\mu_{n_i}\}_{i=1}^\infty$ be an arbitrary subsequence of $\{\mu_n\}_{n=1}^\infty$. Then $\{f(\mu_{n_i})\}_{i=1}^\infty \subseteq (1-\delta, 1+\delta) \subseteq [1-\delta, 1+\delta]$. There is a subsequence $\{\mu_{n_{ij}}\}_{j=1}^\infty$ of $\{\mu_{n_i}\}_{i=1}^\infty$ such that $f(\mu_{n_{ij}}) \to \bar{t} \in [1-\delta, 1+\delta]$ as $j \to \infty$. Since $f(\mu_{n_{ij}})\mu_{n_{ij}} \to \bar{t}\mu$ and hence $\bar{t}\mu \in \Sigma_A$, $\bar{t} \in (1-\delta, 1+\delta)$. Note that

$$\begin{aligned}
\|f(\mu_{n_{ij}})\mu_{n_{ij}} - \lambda\| &= \|f(\mu_{n_{ij}})\mu_{n_{ij}} - (1 \pm \delta)\mu_{n_{ij}} + (1 \pm \delta)\mu_{n_{ij}} - (1 \pm \delta)\lambda + (1 \pm \delta)\lambda - \lambda\| \\
&\leqslant |f(\mu_{n_{ij}}) - (1 \pm \delta)| \, \|\mu_{n_{ij}}\| + (1 \pm \delta)\|\mu_{n_{ij}} - \lambda\| + \delta\|\lambda\| \\
&\leqslant \delta(\|\lambda\|) + (1+\delta)(\varepsilon/2) + \delta\|\lambda\| < 2\delta(\|\lambda\|+1) < \eta.
\end{aligned}$$

Thus mult$(\bar{t}\mu)$ is odd. Thus minimality of $f(\mu)$ implies $\bar{t} \geqslant f(\mu)$. Suppose now $\bar{t} > f(\mu)$. By construction, $B' \cap (1-\delta, f(\mu_{n_{ij}}))\{\mu_{n_{ij}}\} = \varnothing$. Now $f(\mu_{n_{ij}}) \to \bar{t} > f(\mu)$ implies there is $J_0 \in Z^+$ such that $j \geqslant J_0$ implies $f(\lambda_{n_{ij}}) \geqslant f(\lambda) + \sigma$, where $\sigma > 0$ and such that $f(\mu) + \sigma < \bar{t}$. Thus we contradict the fact mult $f(\mu)\mu$ is odd. Hence $\bar{t} = f(\mu)$. Since the subsequence we choose was arbitrary and the limit is unique, f is continuous on C.

We now define a function $\bar{f} : C \to B' \cap \overline{B(\lambda, \eta)}$ by $\bar{f}(\mu) = f(\mu)\mu$. Then $\bar{f}(C)$ is the desired codimension 1 manifold.

Corollary 3.4. If $\lambda \in \Sigma_A$ and mult λ is odd, then λ is not an isolated point in B provided $\dim \Lambda > 1$.

Corollary 3.5. If $\lambda \in B'$, then $\Sigma_A \cap B'$ at λ does not consist solely of a closed, totally disconnected set provided $\dim \Lambda > 1$.

We now give a perturbation theory result which is an aid in establishing a significant consequence of theorem 3.3. Our proof will follow that of [21, theorem II.2.1], making liberal use of ideas from [14]. First, however, we recall some facts from functional analysis needed for the proof.

Let A be a compact linear operator on a real Banach space E and $\lambda \in \mathbb{R}$ an eigenvalue of A with mult $\lambda =$ geomult $\lambda = h$. Then $E = N(A - \lambda I) \oplus R(A - \lambda I)$, where $N(A - \lambda I) = \langle \Phi_1, \ldots, \Phi_h \rangle$ with $\|\Phi_i\| = 1$, $i = 1, 2, \ldots, h$. By the Hahn–Banach Theorem, there is a collection $\{\psi_1, \ldots, \psi_n\} \subseteq E^*$ such that $(\Phi_i, \psi_j) = \delta_{ij}$, for $i, j = 1, 2, \ldots, h$ and $(x, \psi_j) = 0$ for $x \in R(A - \lambda I)$, $j = 1, 2, \ldots, h$, where $(f, g) = g(f)$ for $f \in E$ and $g \in E^*$. Then there is a continuous linear mapping $T: E \to E$ such that $T\Phi_i = 0$ for $i = 1, 2, \ldots, h$, and $T|_{R(A-\lambda I)} = \hat{A}_\lambda$, the pseudoinverse of $A - \lambda I$. It then follows that for any $f \in E$,

$$T(A - \lambda I)f = f - \sum_{i=1}^{h} (f, \psi_i)\, \Phi_i.$$

THEOREM 3.6. Let $\lambda_0 \in \Sigma_A$ with mult $\lambda_0 =$ geomult λ_0. Then there is a nbd V of λ_0 such that for any $\lambda \in V$,

$$\sum_{t\lambda \in \Sigma_A \cap V} \text{geomult}(t\lambda) \leq \text{mult}\, \lambda_0.$$

Proof. The hypothesis is equivalent to the statement that 1 is an eigenvalue of $A(\lambda_0)$ with mult 1 = geomult 1. Note $\lambda_0 \neq 0$, and let X denote the set $\{\mu \in E : \|\mu\| = \|\lambda_0\|\}$. For $\mu \in X$, $A(\mu) = A(\lambda_0 + (\mu - \lambda_0))$. Let $\varepsilon = \mu - \lambda_0$ and define $B(\varepsilon) = A(\lambda_0 + \varepsilon) = A(\mu)$. Then 1 is an eigenvalue for $B(0)$, and if λ is an eigenvalue for $B(\varepsilon)$ with $\varepsilon \in \tilde{X} = X - \lambda_0$ and $\lambda > 0$, then $\lambda^{-1/k}(\lambda_0 + \varepsilon) \in \Sigma_A$ by (2.2). Thus, with no loss of generality, we proceed as follows.

Suppose that the map $\varepsilon \to A(\varepsilon)$ is a continuous map from X to $K(E)$, where X is a codimension 1 manifold passing through the origin of a Banach space Λ. Suppose also that $A(0) = A$ has real eigenvalue λ with mult $\lambda =$ geomult $\lambda = h$. Let T, $\{\Phi_1, \ldots, \Phi_h\}$, and $\{\psi_1, \ldots, \psi_h\}$ be as in the preceding exposition.

Assume now that $A(\varepsilon)$ has perturbed eigenvalue $\lambda(\varepsilon)$ and corresponding eigenvector $\phi(\varepsilon)$. Let $B(\varepsilon) = A(\varepsilon) - A$ and $\mu(\varepsilon) = \lambda(\varepsilon) - \lambda$. Then $(A - \lambda)\phi(\varepsilon) = (\mu(\varepsilon) - B(\varepsilon))\phi(\varepsilon)$. Thus

$$(\phi(\varepsilon), (\mu(\varepsilon) - B(\varepsilon))^*\psi_j) = 0 \tag{3.1}$$

for $j = 1, 2, \ldots, h$. Since

$$\phi(\varepsilon) = T(A - \lambda I)\phi(\varepsilon) + \sum_{i=1}^{h} (\phi(\varepsilon), \psi_i)\, \Phi_i,$$

$\phi(\varepsilon)$ is of the form

$$T(\mu(\varepsilon) - B(\varepsilon))\, \phi(\varepsilon) + \sum_{i=1}^{h} c_i \Phi_i.$$

Let

$$\omega = \sum_{i=1}^{h} c_i \Phi_1 \qquad \text{and} \qquad S = T(\mu(\varepsilon) - B(\varepsilon)).$$

The equation $\phi(\varepsilon) = \omega + S\phi(\varepsilon)$ has solution

$$\sum_{\nu=0}^{\infty} S^\nu \omega$$

provided $\|S\| < 1$. Thus

$$(\phi(\varepsilon), (\mu(\varepsilon) - B(\varepsilon))^* \psi_j) = \sum_{i=1}^{h} c_i \left(\sum_{\nu=0}^{\infty} S^\nu \Phi_i, (\mu(\varepsilon) - B(\varepsilon))^* \psi_j \right).$$

Thus (3.1) is equivalent to

$$\sum_{i=1}^{h} c_i \left((\mu(\varepsilon) - B(\varepsilon)) \sum_{\nu=0}^{\infty} S^\nu \Phi_i, \psi_j \right) = 0 \tag{3.2}$$

$j = 1, 2, \ldots, h$. System (3.2) has a nontrivial solution $\{c_1, \ldots, c_h\}$ if and only if

$$\det\left((\mu(\varepsilon) - B(\varepsilon)) \sum_{\nu=0}^{\infty} S^\nu \Phi_i, \psi_j \right) = 0. \tag{3.3}$$

(3.3) Provides a condition which any eigenvalue for $A(\varepsilon)$ must satisfy. Thus we define

$$f_{ij}(\mu, \varepsilon) = \left(\sum_{\nu=0}^{\infty} (\mu - B(\varepsilon)) [T(\mu - B(\varepsilon))]^\nu \Phi_i, \psi_j \right).$$

Now $B(\varepsilon)$ is continuous in ε with $B(0) = 0$ and T is bounded. Thus

$$\left(\sum_{\nu=0}^{\infty} (\mu - B(\varepsilon)) [T(\mu - B(\varepsilon))]^\nu \Phi_i, \psi_j \right)$$

is a power series in μ for $|\mu|$ small uniformly for ε sufficiently near 0 in X. Furthermore, $f_{ij}(\mu, \varepsilon)$ can be extended to complex valued μ with maintenance of uniformity in ε. The same is true for $F(\mu, \varepsilon) = \det(f_{ij}(\mu, \varepsilon))$.

$$\begin{aligned} \text{Now } f_{ij}(\mu, 0) &= \left(\sum_{\nu=0}^{\infty} \mu [T\mu]^\nu \Phi_i, \psi_j \right) \\ &= \left(\sum_{\nu=0}^{\infty} \mu^{\nu+1} T^\nu \Phi_i, \psi_j \right) \\ &= (\mu \Phi_i, \psi_j) \\ &= \mu \delta_{ij}. \end{aligned}$$

Thus $F(\mu, 0) = \mu^h$. Now $F(\mu, \varepsilon)$ is defined for $\mu \in \mathbb{C}$ such that $|\mu| \leq R_1$ and $\varepsilon \in X$ with $\|\varepsilon\| \leq R_2$ for $R_1 > 0$ and $R_2 > 0$ sufficiently small. Then, as in the proof of lemma I, B3 in [14], we employ Rouché's theorem to show that if $0 < R < R_1$, there is $0 < R' < R_2$ such that for $\varepsilon \in X \cap B(0, R')$, there are exactly h numbers, counting multiplicities, $\mu_1(\varepsilon), \ldots, \mu_h(\varepsilon)$ with $|\mu_i(\varepsilon)| < R$ and $F(\mu_i(\varepsilon), \varepsilon) = 0$, $i = 1, 2, \ldots, h$.

Hence, if we let $\mu(\varepsilon) = \mu_k(\varepsilon)$ for some $k \in \{1, 2, \ldots, h\}$, where $\mu_k(\varepsilon) \in \mathbb{R}$, there exist $c_1(\varepsilon), \ldots, c_h(\varepsilon) \in \mathbb{R}$ with

$$\left\| \sum_{i=1}^{h} c_i(\varepsilon) \Phi_i \right\| = 1$$

and

$$\sum_{i=1}^{h} c_i(\varepsilon)\left(\sum_{\nu=0}^{\infty}(\mu(\varepsilon)-B(\varepsilon))[T(\mu(\varepsilon)-B(\varepsilon))]^{\nu}\Phi_i, \psi_j\right)=0, \qquad j=1,2,\ldots,h.$$

Now define

$$\phi(\varepsilon)=\sum_{\nu=0}^{\infty}[T(\mu(\varepsilon)-B(\varepsilon))]^{\nu}\left(\sum_{i=1}^{h}c_i(\varepsilon)\Phi_i\right).$$

Then

$$\phi(\varepsilon)=\sum_{i=1}^{h}c_i(\varepsilon)\Phi_i+T(\mu(\varepsilon)-B(\varepsilon))\,\phi(\varepsilon)$$

and

$$((\mu(\varepsilon)-B(\varepsilon))\,\phi(\varepsilon),\psi_j)=0, \qquad j=1,2,\ldots,h.$$

Hence

$$(A-\lambda)\,\phi(\varepsilon)=(A-\lambda)\left[\sum_{i=1}^{h}c_i(\varepsilon)\Phi_i+T(\mu(\varepsilon)-B(\varepsilon))\,\phi(\varepsilon)\right]=(\mu(\varepsilon)-B(\varepsilon))\,\phi(\varepsilon).$$

Thus $A(\varepsilon)\,\phi(\varepsilon)=(\lambda+\mu(\varepsilon))\,\phi(\varepsilon)$. Furthermore

$$\phi(\varepsilon)=\sum_{i=1}^{h}c_i(\varepsilon)\Phi_i+\sum_{\nu=1}^{\infty}[T(\mu(\varepsilon)-B(\varepsilon))]^{\nu}\left(\sum_{i=1}^{h}c_i(\varepsilon)\Phi_i\right)$$

implies that $\|\varepsilon\|$ may be chosen sufficiently small to guarantee

$$\left\|\phi(\varepsilon)-\sum_{i=1}^{h}c_i(\varepsilon)\Phi_i\right\|<\tfrac{1}{2}.$$

Hence $\phi(\varepsilon)\neq 0$. Thus $\phi(\varepsilon)$ is an eigenfunction for $A(\varepsilon)$ corresponding to eigenvalue $\lambda+\mu(\varepsilon)$.

We now have the following important consequence of theorem 3.5 and theorem 3.3.

THEOREM 3.7. Suppose $\lambda\in\Sigma_A$ with mult $\lambda=1$. Then Σ_A at λ is a codimension 1 manifold each point μ of which has mult $\mu=1$.

Remark 3.8. Theorem 3.7 may be established via the implicit function theorem provided the map $\lambda\to A(\lambda)$ is differentiable. (2.2) and theorems 3.3 and 3.6 allow the removal of the differentiability assumption.

4. AN APPLICATION TO NONLINEAR STURM–LIOUVILLE BOUNDARY VALUE PROBLEMS

In this section we consider the boundary value problem

$$-(p(\tau)x'(\tau))'+q(\tau)x(\tau)=r(\tau)x(\tau)+f(\tau,x(\tau),x'(\tau)), \tag{4.1}$$

where $\tau \in [a, b]$ and x is to satisfy

$$\alpha x(a) + \alpha' x'(a) = 0 \tag{4.2.i}$$

$$\beta x(b) + \beta' x'(b) = 0. \tag{4.2.ii}$$

We assume that p is positive and continuously differentiable on $[a, b]$; q is continuous on $[a, b]$; r is positive and continuous on $[a, b]$; $f : [a, b] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and satisfies $f(\tau, u, v) = o(|u| + |v|)$ uniformly for $\tau \in [a, b]$; and $(|\alpha| + |\alpha'|)(|\beta| + |\beta'| > 0$.

We investigate the solution set to (4.1)–(4.2) viewed as a bifurcation problem with the coefficient functions p and q varying as parameters (see [2]). We begin by showing that (4.1)–(4.2) may be considered in the context of Section 2.

Let $L(p, q)$ be defined by $L(p, q)x = -(px')' + qx$, where the independent variable τ has been suppressed. If $t > 0$, it is clear that $L(tp, tq) = tL(p, q)$. Furthermore, since solutions to initial value problems for

$$L(p, q)x(\tau) = 0, \tag{4.3}$$

$\tau \in (a, b)$, are unique, solutions to initial value problems for (4.3) depend continuously (in fact, analytically) on p and q. The same is then true for the Green's functions which are associated with (4.3)–(4.2).

Let $V = \{u \in C^1[a, b] : u > 0$ on $[a, b]\} \times C[a, b]$. Then V is open $C^1[a, b] \times C[a, b]$ and $tV \subseteq V$ for $t > 0$. Now let $D = \{(p, q) \in V : L(p, q)x = 0$ and x satisfies (4.2) imply $x \equiv 0\}$. The preceeding observations imply that D is open. Furthermore it is easy to see that $tD \subseteq D$.

Let $(p, q) \in D$. Then the operator $G(p, q)$, given by

$$G(p, q)x(\tau) = \int_a^b g(p, q)[\tau, s]r(s)x(s)\, ds, \tag{4.4}$$

where $g(p, q) : [a, b] \times [a, b] \to \mathbb{R}$ is the Green's function for (4.3)–(4.2) at (p, q), is a compact linear operator on the real Banach space E, given as the continuously differentiable functions on $[a, b]$ which satisfy (4.2). Observe that if $x \in E$ and $t > 0$, $L(tp, tq)G(tp, tq)x(\tau) = r(\tau)x(\tau)$. Hence $L(p, q)[tG(tp, tq)x(\tau) - G(p, q)x(\tau)] = 0$ and therefore $G(tp, tq) = t^{-1} G(p, q)$ for $t > 0$.

Using $g(p, q)$ for $(p, q) \in D$, (4.1)–(4.2) is equivalent to

$$x = G(p, q)x + F(p, q, x), \tag{4.5}$$

for $x \in E$, where $F(p, q, x)$ is given by

$$F(p, q, x)(\tau) = \int_a^b g(p, q)[\tau, s]f(s, x(s))\, ds. \tag{4.6}$$

Since the map $(p, q) \to g(p, q)$ is continuous (in fact, analytic) and $f(\tau, u, v) = o(|u| + |v|)$ uniformly for $\tau \in [a, b]$, the results of Section 2 are applicable to (4.5). In particular, (2.2) holds with $k = -1$.

Suppose now $(p, q) \in D$. It follows from linear Sturm–Liouville theory [11] that there are sequences $\{x_n\}_{n=1}^{\infty} \subseteq E$, $x_n \neq 0$, and $\{\lambda_n\}_{n=1}^{\infty} \subseteq \mathbb{R}$, $\lambda_n = \lambda_n(p, q)$, such that $\lambda_n \to +\infty$ as $n \to +\infty$ and $L(p, q)x_n = \lambda_n r x_n$. Furthermore, λ_n is a simple eigenvalue for the linear boundary value problem. The following result is therefore a consequence of theorem 2.1.

THEOREM 4.1. Consider the nonlinear boundary value problem (4.1)–(4.2) (or, equivalently, operator equation (4.5)). For each $(p, q) \in D$, there is a sequence $\{\lambda_n(p, q)\}_{n=1}^{\infty}$ of real numbers with $\lambda_n(p, q) \to +\infty$ such that

$$\left\{\frac{1}{\lambda_n(p, q)}(p, q)_{n \geq k(p,q)}\right\} \subseteq B' \cap D,$$

where $k(p, q) = \min\{n : \lambda_n(p, q) > 0\}$. Furthermore, the map $(p, q) \to k(p, q)$ is continuous and so constant on components of D.

Theorem 3.7 insures that B is a codimension 1 manifold at each such point $\lambda_n^{-1}(p, q)(p, q)$. However, we may make this result more explicit, proceeding as follows.

Let $D_0 = \{(p, q) \in D : \|(p, q)\|_{C^1[a,b] \times C[a,b]} = 1\}$ and for $i \geq 1$, let $D_i = \{(p, q) \in D : \text{if } x \in E, x \neq 0 \text{ and } L(p, q)x = rx, \text{ then } x \text{ has } i-1 \text{ simple zeros in } (a, b)\}$. Consider a component $\tilde{D}_0$ of D_0 and let $k(\tilde{D}_0)$ be the value of the map $(p, q) \to k(p, q)$ on $\tilde{D}_0$. B is then characterized as follows.

THEOREM 4.2. Let $i \geq k(\tilde{D}_0)$. Then the map ψ_i, given by

$$\psi_i(p, q) = \frac{1}{\lambda_i(p, q)}(p, q)$$

is analytic map from $\tilde{D}_0$ into D_i.

Proof. The result may be obtained via theorem II.5.16 and ideas in Section IV.3.5 of [17]. See also [2].

We now turn our attention to the nontrivial solutions which emanate from B. Our result is:

THEOREM 4.3. Let $(p_0, q_0) \in \psi_i(\tilde{D}_0)$, where $\tilde{D}_0$ is a component of D_0. Let $h : \mathbb{R} \to D$ be a proper crossing of changing degree at (p_0, q_0) such that $h(\mathbb{R}) \cap \psi_i(\tilde{D}_0) = \{(p_0, q_0)\}$. Then there is a continuum $\mathscr{C}$ of nontrivial solutions to (4.5) such that $\mathscr{C}$ meets $\partial[(\mathbb{R}^+ \tilde{D}_0) \times E]$ and such that if $((p, q), x) \in \mathscr{C}$, then x has $i - 1$ simple zeros in (a, b).

Proof. Apply theorem 2.5 along with the Schmitt–Smith lemma [22, theorem 2.5].

Remarks 4.4. (i). By placing additional assumptions on f (see, for example, [12] and [19]), we may obtain the existence of nontrivial solutions with $i - 1$ simple zeros in (a, b) for each $(p, q) \in (0, 1)\psi_i(D_0)$. Further assumptions on f may be made to guarantee that $\|(p, q, x)\|_{D \times E} \to +\infty$ as $\|(p, q)\|_D \to 0$. (ii). The main results of [2] show that the continua of theorem 4.3 are subsets of solution continua which are infinite dimensional at every point.

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